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ABSTRACT

This study shows that the test statistic for Analysis of Covariance (ANCOVA) has a noncentral F-districution with noncentrality parameter equal to zero if and only if the regression planes are homogeneous and/or the vector of overall covariate means is the null vector. The effect of heterogeneous regression slope parameters is to either increase or decrease the power of the F-test of ANCOVA. This depends on the relationship of the true difference of the treatment effects of two groups with respect to the dot product of the vector of overall covariate means and the vector of the difference of the two groups' slope parameters. (Author/RC)

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AN ANALYTICAL INVESTIGATION OF THE ROBUSTNESS AND POWER OF ANCOVA WITH THE PRESENCE OF HETEROGENEOUS REGRESSION SLOPES.*

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General discussions of the misuse of the analysis of covariance statistical procedure may be found in articles by Elashoff (1969) and Glass, Peckham and Sanders (1972). Two problems which are discussed in these references are the robustness and the power of the F-test of the analysis of covariance with the presence of heterogeneous regression slopes. Both of the problems may be addressed by application of a K-sample regression linear model.

A model equation for a one-way classification analysis of covariance procedure is

$$\underset{\sim}{y}_k = \underset{\sim}{1}_{n_k} (\mu + \tau_k - \underset{\sim}{\overline{P}}\overset{\circ}{\underset{\sim}{\beta}}_k) + \underset{\sim}{P}_k\overset{\circ}{\underset{\sim}{\beta}}_k + \underset{\sim}{e}_k ,$$

where τ_k is the treatment effect (fixed) in group k, $\underset{\sim}{P}_k$ is the matrix of observed covariates in group k, $\underset{\sim}{\overline{P}}$ is the 1xp row vector of covariate means taken over the K samples and k = 1, 2, . . . , K.

Let $\alpha_k = \mu + \tau_k - \underset{\sim}{\overline{P}}\overset{\circ}{\underset{\sim}{\beta}}_k$, then

$$\underset{\sim}{y}_k = \underset{\sim}{1}_{n_k}\alpha_k + \underset{\sim}{P}_k\overset{\circ}{\underset{\sim}{\beta}}_k + \underset{\sim}{e}_k . \tag{1}$$

The analysis of covariance procedure tests the hypothesis H_0: $\tau_1 = \tau_2 = \ldots = \tau_k$ by testing the hypothesis H_1: $\alpha_1 - \alpha_2 = 0$, $\alpha_2 - \alpha_3 = 0, \ldots, \alpha_{K-1} - \alpha_K = 0$. The hypotheses H_0 and H_1 are

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equivalent if $\overset{\circ}{\beta}_1 = \overset{\circ}{\beta}_2 = \ldots = \overset{\circ}{\beta}_k$ since

$$\alpha_i - \alpha_{i+1} = \tau_i - \tau_{i+1} + \overline{P}(\overset{\circ}{\beta}_{i+1} - \overset{\circ}{\beta}_i) .$$

Note also H_o and H_1 are equivalent if $\overline{P} = 0'$.

Equation 1 has the form of a K-sample regression model and H_o: $\tau_1 = \tau_2 = \ldots = \tau_k$ is tested by testing the equality of the intercept parameters α_k, $k = 1, 2, \ldots K$, of the model. Thus the treatment effects are tested at the points where the regression planes defined by each group intersect the criterion variable axis.

Atiqullah (1964) used analytic methods to investigate the effect of treatment slope interaction in analysis of covariance with one covariate. His investigation showed that the expected value of the difference of two estimated treatment effects is biased when homogeneity of regression is not tenable.

Peckham's empirical study (1970), based on Atiqullah's work, suggested that the F-test of the analysis of covariance is increasingly conservative as the variance of the within group regression slope parameter increased. His result differs from the one to be developed here, because his design chose the unequal regression lines to intersect at the dependent variable intercept, the point at which treatment effects are tested.

Kocher (1974) and Hamilton (1974) independently considered the same problem in the case when the covariate was a random variable. In both studies, the computer-generated values of the covariate yielded a population grand mean of zero for the covariate, thereby creating nearly the same situation as in Peckham's study.

These three empirical studies found that, in general, the probability of rejecting the true null hypothesis of equal treatment effects decreased when the separate group regression lines are not parallel. One should guard against over generalizing from the results of these investigations for the following reasons: (1) The investigators had problems with simulating the linear model for analysis of covariance. These problems included unequal group error variances and non-zero correlations between the covariate and the error variable. (2) Each of the investigators made certain assumptions about the linear model. The assumption that the grand mean of the covariate is zero has a definite effect on the outcome of this problem. And, (3) each study dealt with only one concomitant variable.

Analytical Investigation of Robustness Using the K-Sample Regression Linear Model

A K-sample regression linear model will allow an analytical study of the effect of heterogeneous regression planes to the analysis of covariance procedure.

The model equation of analysis of covariance, Equation 1 of the previous section, may be incorporated into the linear model

$$\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{\beta} + \underset{\sim}{e} \qquad (2)$$

where

$$\underset{\sim}{y} = (\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_k)',$$

$$\underset{\sim}{\beta} = (\alpha_1, \overset{\circ}{\underset{\sim}{\beta}}_1, \alpha_2, \overset{\circ}{\underset{\sim}{\beta}}_2, \ldots, \alpha_k, \overset{\circ}{\underset{\sim}{\beta}}_k)', \qquad (3)$$

$$\underset{\sim}{e} = (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_k)',$$

and

$$\underset{\sim}{X} = \mathrm{DIAG}[(\underset{\sim}{1}_{N_1} : \underset{\sim}{P}_1) , (\underset{\sim}{1}_{N_2} : \underset{\sim}{P}_2) , \ldots , (\underset{\sim}{1}_{N_K} : \underset{\sim}{P}_K)] ,$$

i. e. $\underset{\sim}{X}$ is a super diagonal matrix.

The analysis of covariance hypothesis H_1: $\alpha_1 - \alpha_2 = 0$, $\alpha_2 - \alpha_3 = 0$, $\ldots$, $\alpha_{K-1} - \alpha_K = 0$ may be written as H_1: $\underset{\sim}{L}'\underset{\sim}{\beta} = 0$ where

$$\underset{\sim}{L}' = \begin{bmatrix} 1 & \underset{\sim}{0}' & -1 & \underset{\sim}{0}' & 0 & \ldots & 0 & \underset{\sim}{0}' & 0 & \underset{\sim}{0}' \\ 0 & \underset{\sim}{0}' & 1 & \underset{\sim}{0}' & -1 & & 0 & \underset{\sim}{0}' & 0 & \underset{\sim}{0}' \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & \underset{\sim}{0}' & 0 & \underset{\sim}{0}' & 0 & \ldots & 1 & \underset{\sim}{0}' & -1 & \underset{\sim}{0}' \end{bmatrix} . \qquad (4)$$

$\underset{\sim}{L}'$ is a (K-1) x K(p+1) matrix of full row rank and

$$\underset{\sim}{L}'\underset{\sim}{\beta} = \begin{bmatrix} \tau_1 - \bar{\underset{\sim}{P}}\mathring{\underset{\sim}{\beta}}_1 - \tau_2 + \bar{\underset{\sim}{P}}\mathring{\underset{\sim}{\beta}}_2 \\ \tau_2 - \bar{\underset{\sim}{P}}\mathring{\underset{\sim}{\beta}}_2 - \tau_3 + \bar{\underset{\sim}{P}}\mathring{\underset{\sim}{\beta}}_3 \\ \cdot \\ \cdot \\ \cdot \\ \tau_{K-1} - \bar{\underset{\sim}{P}}\mathring{\underset{\sim}{\beta}}_{K-1} - \tau_K + \bar{\underset{\sim}{P}}\mathring{\underset{\sim}{\beta}}_K \end{bmatrix} . \qquad (5)$$

If $\mathring{\underset{\sim}{\beta}}_1 = \mathring{\underset{\sim}{\beta}}_2 = \ldots = \mathring{\underset{\sim}{\beta}}_K$, then

$$\underset{\sim}{L}'\underset{\sim}{\beta} = [\tau_1 - \tau_2 , \tau_2 - \tau_3 , \ldots , \tau_{K-1} - \tau_K]'$$

and $\underset{\sim}{L}'\underset{\sim}{\beta} = 0$ if $\tau_1 = \tau_2 = \tau_3 = \ldots = \tau_K$, with the premise of homogeneity of regression.

Suppose we rewrite Equation 5 as

$$L'\beta = \begin{bmatrix} (\tau_1 - \tau_2) + \overline{P}(\mathring{\beta}_2 - \mathring{\beta}_1) \\ (\tau_2 - \tau_3) + \overline{P}(\mathring{\beta}_3 - \mathring{\beta}_2) \\ \vdots \\ (\tau_{K-1} - \tau_K) + \overline{P}(\mathring{\beta}_K - \mathring{\beta}_{K-1}) \end{bmatrix}$$

and let $\gamma_k = \overline{P}(\mathring{\beta}_{k+1} - \mathring{\beta}_k)$, $k = 1, 2, \ldots, K - 1$. Define

$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{K-1})'$. Note that if $\overline{P} = 0'$, then $\gamma = 0$. Also let

$\Delta = [(\tau_1 - \tau_2) , (\tau_2 - \tau_3) , \ldots, (\tau_{K-1} - \tau_K)]'$, then $L'\beta = \Delta + \gamma$.

Now if $e: N(0, \sigma^2 I_N)$ then, the F-statistic for testing $H_1: L'\beta = 0$ is

$$F(H_1) = \frac{(L'\hat{\beta})' [L'(X'X)^{-1}L]^{-1} (L'\hat{\beta})/(K-1)}{SSE/(N-Kp-K)} ,$$

where $SSE = y' [I-X(X'X)^{-1}X'] y$. $F(H_1)$ has non-central F-distribution

with non-centrality parameter $\lambda = \beta'L [L'(X'X)^{-1}L]^{-1} L'\beta/2\sigma^2$, denoted

$F(H_1): F(K-1, N-Kp-K, \lambda)$.

Let $A = [L'(X'X)^{-1}L]^{-1}/2\sigma^2$, then $\lambda = (\Delta + \gamma)' A(\Delta + \gamma)$ or

$$\lambda = \Delta'A\Delta + \gamma'A\gamma + 2\gamma'A\Delta . \tag{6}$$

Suppose the treatment effects are equal, $\Delta = 0$, then $\lambda = \gamma'A\gamma$. Since A is a positive definite matrix λ is greater than or equal to zero, and equals zero if and only if $\gamma = 0$, i.e. when the regression planes are homogeneous or when $\overline{P} = 0'$.

Therefore $F(H_1)$ has a central F-distribution when the hypothesis H_1 is true only if $\underset{\sim}{\gamma} = \underset{\sim}{0}$. In general when $\overline{\underset{\sim}{P}} \neq \underset{\sim}{0}'$, a consequence of violating the homogeneity of regressions assumption when using the analysis of covariance procedure is to increase the probability of a Type I error, rejecting H_1 when H_1 is true. This follows from the fact that the expected value of a non-central F-statistic is larger than that of a corresponding central F-statistic. That is, $E[F(H_1)] = n_2(1 + 2\lambda/n_1)/(n_2 - 2)$ when $F(H_1)$ is a non-central F-statistic with n_1 and n_2 degrees of freedom and non-centrality parameter λ. If $\lambda = 0$, then $E[F(H_1)] = n_2/(n_2 - 2)$ (see Searle, 1971, p. 51).

There is a shortcoming with the above approach which does affect the outcome to some extent. The model above is not the usual analysis of covariance model. The usual model estimates the within group slope parameters, $\overset{\circ}{\underset{\sim}{\beta}}$, by pooling the observed covariate data from all the treatment groups while the above model estimates the within group slope parameters K times, using the observed covariate data within each treatment group.

The model equation of a one-way classification analysis of covariance with the premise of homogeneity of regression, in the form of Equation 2, is

$$\underset{\sim}{y} = \underset{\sim}{X}^*\underset{\sim}{\beta}^* + \underset{\sim}{e}^* ,$$

where $\underset{\sim}{y}$ is as before, $\underset{\sim}{e}^*$ has the same distribution as $\underset{\sim}{e}$,

$$\underset{\sim}{\beta}^* = (\overset{\circ}{\underset{\sim}{\beta}}, a_1^*, a_2^*, \ldots , a_k^*)' ,$$

and

$$\mathbf{X}^* = \begin{pmatrix} \mathbf{P}_1 & \mathbf{1}_{N_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{P}_2 & \mathbf{0} & \mathbf{1}_{N_2} & \cdots & \mathbf{0} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{P}_K & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_{N_K} \end{pmatrix}$$

For this model the unbiased estimate of σ^2 is $\hat{\sigma}^2 = \mathbf{y}' \, [\mathbf{I}-\mathbf{X}^*(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}] \, \mathbf{y}/(N-K-p)$, while, for the model used in the previous development, the unbiased estimate of σ^2 is

$$\hat{\hat{\sigma}}^2 = \mathbf{y}' \, [\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'] \, \mathbf{y}/(N-Kp-K) \, .$$

Also the usual analysis of covariance procedure estimates α_k^* by $\hat{\alpha}_k^* = \bar{y}_k - \bar{\mathbf{P}}_k\hat{\hat{\boldsymbol{\beta}}}$ while the model in this study estimates α_k by $\alpha_k = \bar{y}_k - \bar{\mathbf{P}}_k\hat{\hat{\boldsymbol{\beta}}}_k$.

The differences between the two models can only lead to the conclusion that the models are equivalent for large samples given reasonable values of K and p. Theoretically the two models are equivalent, given homogeneity of regression, as n_k approaches infinity.

However, the conclusions of an increased probability of committing a Type I error when homogeniety of regression is not tenable still stands. For this is an optimum result given a large sample and in statistics results are usually less than optimum when based on small samples.

Power of the Analysis of Covariance Test of Hypothesis

The model developed in the previous section may also be used to investigate the power of the analysis of covariance test of the hypothesis of equal treatment effects.

The power of a test is the probability of rejecting the null hypothesis when it is not true or the complement of the probability of committing a Type II error.

The consideration of this topic, using the present approach, is of little practical value since one usually uses a power function to determine a minimum sample size for which a prespecified practical difference between treatments would be judged statistically significant. Considering the comments at the end of the previous section, the model we shall use will yield optimum results for large samples only. But the importance of this topic can be summed up in two sentences found in an article by Glass, Peckham and Sanders (1973, pp. 277-9). They wrote, "Further study is needed to determine the effects of unequal slopes on the power of analysis of covariance. This could very well be the crucial issue."

The null hypothesis of analysis of covariance is $H_1: \underset{\sim}{L}'\underset{\sim}{\beta} = \underset{\sim}{0}$ where $\underset{\sim}{\beta}$ and L' are defined in the previous section by Equations 3 and 4 respectively. The test statistic of H_1 is

$$F(H_1) = \frac{(\underset{\sim}{L}'\hat{\underset{\sim}{\beta}})' \, [\underset{\sim}{L}'(\underset{\sim}{X}'\underset{\sim}{X})^{-} \underset{\sim}{L}]^{-1} \, (\underset{\sim}{L}'\hat{\underset{\sim}{\beta}})/(K-1)}{SSE/(N-Kp-K)}$$

and $F(H_1)$: $F(K-1,\ N-Kp-K)$ when H_1 is true. If H_1 is not true, then $F(H_1)$: $F'(K-1,\ N-Kp-K,\ \lambda)$, where $\lambda = (\underset{\sim}{L}'\underset{\sim}{\beta})' \, [\underset{\sim}{L}'(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{L}]^{-1} \, (\underset{\sim}{L}'\underset{\sim}{\beta})/2\sigma^2$.

Let P(II) be the probability of a Type II error occurring, then $\beta(\lambda) = 1 - P(II)$ is, by definition, the power of the test evaluated at the point λ. We shall refer to $\beta(\lambda)$ as the power function.

Now $P(II) = \Pr\{F(H_1) \leq F_{1-\alpha}(K-1, N-Kp-K)\}$ when $F(H_1)$: $F'(K-1, N-Kp-K, \lambda)$, or

$$P(II) = \Pr\{F'(K-1, N-Kp-K, \lambda) \leq F_{1-\alpha}(K-1, N-Kp-K)\}$$

where α is the prescribed level of significance of the test. Thus

$$\beta(\lambda) = \Pr\{F'(K-1, N-Kp-K, \lambda) > F_{1-\alpha}(K-1, N-Kp-K)\}$$

(see Searle, 1971, p. 126).

From Equation 6 of the previous section

$$\lambda = \underset{\sim}{\Delta}'\underset{\sim}{A}\underset{\sim}{\Delta} + \underset{\sim}{\gamma}'\underset{\sim}{A}\underset{\sim}{\gamma} + 2\underset{\sim}{\gamma}'\underset{\sim}{A}\underset{\sim}{\Delta}$$

where $\underset{\sim}{\Delta} = (\tau_1 - \tau_2, \tau_2 - \tau_3, \ldots, \tau_{K-1} - \tau_K)'$

$\underset{\sim}{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{K-1})'$, $\gamma_k = \overline{\underset{\sim}{P}}(\overset{\circ}{\underset{\sim}{\beta}}_{k+1} - \overset{\circ}{\underset{\sim}{\beta}}_k)$

and, $\underset{\sim}{A} = [\underset{\sim}{L}'(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{L}]^{-1}/2\sigma^2$.

If $\underset{\sim}{\gamma} = \underset{\sim}{0}$ and $\underset{\sim}{\Delta} = \underset{\sim}{0}$, then $\lambda = \underset{\sim}{0}$ and

$$\beta(\lambda) = \Pr\{F'(K-1, N-Kp-K, 0) > F_{1-\alpha}(K-1, N-Kp-K)\}$$

or $\beta(\lambda) = \alpha$, and thus the test of the hypothesis H_1: $\underset{\sim}{L}'\underset{\sim}{\beta} = \underset{\sim}{0}$ is a level α test.

If $\underset{\sim}{\gamma} = \underset{\sim}{0}$, then $\lambda = \underset{\sim}{\Delta}'\underset{\sim}{A}\underset{\sim}{\Delta}$ and $\lambda > 0$ if $\underset{\sim}{\Delta} \neq \underset{\sim}{0}$.

If $\underset{\sim}{\Delta} = \underset{\sim}{0}$, then $\lambda = \underset{\sim}{\gamma}'\underset{\sim}{A}\underset{\sim}{\gamma}$ and $\lambda > 0$ if $\underset{\sim}{\gamma} \neq \underset{\sim}{0}$.

When $\underset{\sim}{\Delta} \neq \underset{\sim}{0}$ and $\underset{\sim}{\gamma} \neq \underset{\sim}{0}$, then $\lambda \geq 0$ and $\lambda = 0$ only when $\underset{\sim}{\Delta} = -\underset{\sim}{\gamma}$ since $\lambda = (\underset{\sim}{\Delta} + \underset{\sim}{\gamma})'\underset{\sim}{A}(\underset{\sim}{\Delta} + \underset{\sim}{\gamma})$ and the matrix $\underset{\sim}{A}$ is positive definite. Thus when $(\tau_1 - \tau_2, \tau_2 - \tau_3, \ldots, \tau_{K-1} - \tau_K) = -\overline{\underset{\sim}{P}}(\overset{\circ}{\underset{\sim}{\beta}}_2 - \overset{\circ}{\underset{\sim}{\beta}}_1, \overset{\circ}{\underset{\sim}{\beta}}_3 - \overset{\circ}{\underset{\sim}{\beta}}_2, \ldots, \overset{\circ}{\underset{\sim}{\beta}}_K - \overset{\circ}{\underset{\sim}{\beta}}_{K-1})$, the power function equals α.

Thus to restate the conclusion of the previous section, when $\underset{\sim}{\gamma} = \underset{\sim}{0}$, the power function is an increasing function of $\lambda = \underset{\sim}{\Delta}'\underset{\sim}{A}\underset{\sim}{\Delta}$ and the minimum of the power function occurs at the point $\underset{\sim}{\Delta} = \underset{\sim}{0}$. However, when $\underset{\sim}{\gamma} \neq \underset{\sim}{0}$, the minimum of the power function now occurs at the point for which $\lambda = (\underset{\sim}{\Delta} + \underset{\sim}{\gamma})' \underset{\sim}{A}(\underset{\sim}{\Delta} + \underset{\sim}{\gamma}) = 0$. So that $\beta(\lambda) \geq \alpha$ and equals α if and only if $\underset{\sim}{\Delta} = -\underset{\sim}{\gamma}$. Hence, barring the rare event of $\underset{\sim}{\Delta} = -\underset{\sim}{\gamma}$, the level of significance of H_1 will always be greater than its prescribed value of α, when $\underset{\sim}{\gamma} \neq \underset{\sim}{0}$.

How the power function when $\underset{\sim}{\gamma} = \underset{\sim}{0}$ differs from the power function when $\underset{\sim}{\gamma} = \underset{\sim}{a} \neq 0$ depends on $\underset{\sim}{\Delta}$. Let $\lambda_1 = \underset{\sim}{\Delta}'\underset{\sim}{A}\underset{\sim}{\Delta}$ and $\lambda_2 = (\underset{\sim}{\Delta} + \underset{\sim}{a})' \underset{\sim}{A}(\underset{\sim}{\Delta} + \underset{\sim}{a})$, then $\lambda_2 - \lambda_1 = \underset{\sim}{a}'\underset{\sim}{A}\underset{\sim}{a} + 2\underset{\sim}{a}'\underset{\sim}{A}\underset{\sim}{\Delta}$ and $\beta(\lambda_1) < \beta(\lambda_2)$ if $\lambda_1 < \lambda_2$. Note again that $\beta(\lambda_2) > \alpha$ since $\lambda_2 - \lambda_1 > 0$ when $\underset{\sim}{\Delta} = \underset{\sim}{0}$. Let $f(\underset{\sim}{\Delta}) = \lambda_2 - \lambda_1$ or $f(\underset{\sim}{\Delta}) = \underset{\sim}{a}'\underset{\sim}{A}(\underset{\sim}{a} + 2\underset{\sim}{\Delta})$, then

$$f(\underset{\sim}{a}) = 3\underset{\sim}{a}'\underset{\sim}{A}\underset{\sim}{a} ,$$

$$f(-\underset{\sim}{a}/2) = 0 ,$$

$$f(\underset{\sim}{0}) = \underset{\sim}{a}'\underset{\sim}{A}\underset{\sim}{a} ,$$

and

$$f(-\underset{\sim}{a}) = -\underset{\sim}{a}'\underset{\sim}{A}\underset{\sim}{a} .$$

Thus

$$\beta(\lambda_1) < \beta(\lambda_2) \text{ when } \underset{\sim}{\Delta} = \underset{\sim}{a} \text{ or } \underset{\sim}{\Delta} = \underset{\sim}{0} ,$$

$$\beta(\lambda_1) = \beta(\lambda_2) \text{ when } \underset{\sim}{\Delta} = -\underset{\sim}{a}/2 ,$$

and

$$\beta(\lambda_1) > \beta(\lambda_2) \text{ when } \underset{\sim}{\Delta} = -\underset{\sim}{a} .$$

Unless K = 2, it is difficult to determine the relationship between $\beta(\lambda_1)$ and $\beta(\lambda_2)$ when $\underset{\sim}{\Delta}$ is not a constant multiple of $\underset{\sim}{a}$. When K = 2, then $\underset{\sim}{\Delta}$ and $\underset{\sim}{a}$ are scalars.

The further development of this topic will only consider the situation when K = 2. This does not substantially limit the investigation, because even though a is a scalar, it still is a function of p-covariates and now our concern can be considered as a multiple pairwise comparison problem of treatment effects when $K > 2$.

Now $f(\Delta) = aA(a + 2\Delta)$ is a linear equation in Δ and for $\Delta < -a/2$, $f(\Delta) < 0$; for $\Delta > a/2$, $f(\Delta) > 0$; and when $\Delta = -a/2$, $f(\Delta) = 0$.

With this information a graph of $\beta(\lambda_1)$ and $\beta(\lambda_2)$ versus Δ can be drawn and would have the appearance of the following figure.

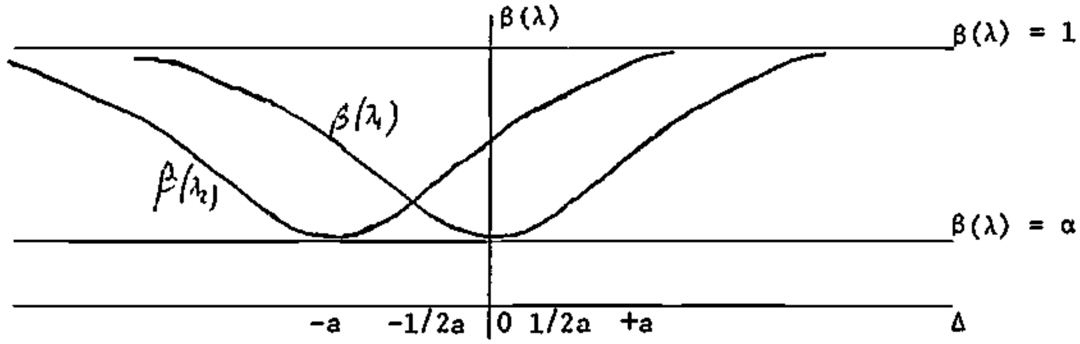

Figure 1

$\beta(\lambda_1)$ is the power function when $\gamma = 0$

$\beta(\lambda_2)$ is the power function when $\gamma = a$

The curves $\beta(\lambda_1)$ and $\beta(\lambda_2)$ are asymptotic to the line $\beta(\lambda) = 1$ and achieve their minimum at $\beta(\lambda) = \alpha$, the level of significance.

Recalling that $\Delta = \tau_1 - \tau_2$ and $a = \bar{P}(\mathring{\beta}_2 - \mathring{\beta}_1)$, if $\tau_1 - \tau_2 < -(1/2)\ \bar{P}(\mathring{\beta}_2 - \mathring{\beta}_1)$, then the power of the F-test of the hypothesis H_0: $\tau_1 = \tau_2$ is reduced when $\mathring{\beta}_1 \neq \mathring{\beta}_2$. But the power of the test is increased when $\tau_1 - \tau_2 > -(1/2)\ \bar{P}(\mathring{\beta}_2 - \mathring{\beta}_1)$ under the same condition.

Thus, to summarize, in the general case when the grand mean vector of the covariates is not zero, heterogeneous regression slopes increase the probability of a Type I error, rejecting H_0 when H_0 is true, and either increases or decreases the power of the F-test of equal treatment effects depending upon the relationship of $(\tau_1 - \tau_2)$ with respect to $\underset{\sim}{\bar{P}}(\underset{\sim}{\mathring{\beta}}_2 - \underset{\sim}{\mathring{\beta}}_1)$. For a sufficiently large difference of $(\tau_1 - \tau_2)$ the power function is close to one, regardless of the value of $\underset{\sim}{\bar{P}}(\underset{\sim}{\mathring{\beta}}_2 - \underset{\sim}{\mathring{\beta}}_1)$.

This result is limited to the case of a large sample experiment with p-covariates and where interest would be in pairwise comparisons of the treatment effects when the number of treatment groups are greater than two.

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